UTOPIAE

Optimisation and Uncertainty Quantification CPD for Teachers of Advanced Higher Physics and Mathematics

Outreach Material

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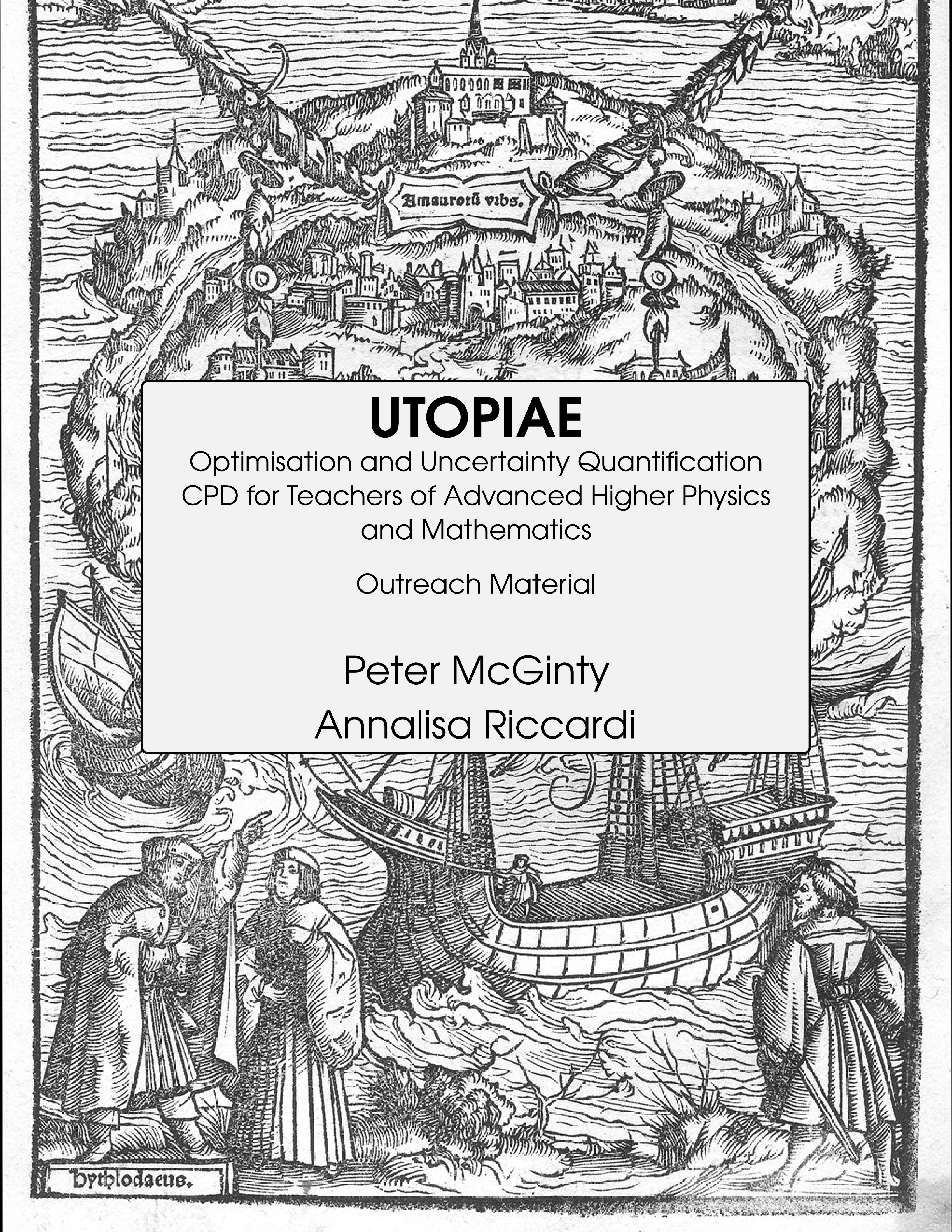

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1. Introduction

1.1 Who is this for?

The UTOPIAE Network is committed to creating high quality engagement opportunities for the Early Stage Researchers working within the network and also outreach materials for the wider academic community to benefit from.

Optimisation and Uncertainty Quantification, whilst representing the future for a large number of research fields, are relatively unknown disciplines and yet the benefit and impact they can provide for researchers is vast. In order to try to raise awareness, UTOPIAE has created a Continuous Professional Development online resource and training sessions aimed at teachers of Advanced Higher Physics and Mathematics, which is in line with Scotland's Curriculum for Excellence. Supported by the Glasgow City Council, the STEM Network, and Scottish Schools Education Resource Centre these materials have been published online for teachers to use within a classroom context. A training workshop run by Early Stage Researchers from the UTOPIAE Network will be hosted at the University of Strathclyde in 2019 and pre-registrations can be made at this link: http://utopiae.eu/2-2/utopiae-training/outreach/optimisation-and-uncertainty-quantification-cpd-training-course/

1.2 What are Optimisation and Uncertainty Quantification and why are they important?

In an expanding world with limited resources and increasing complexity, optimisation and computational intelligence have become a necessity. Optimisation can turn a problem into a solution and computational intelligence can offer new solutions to effectively make complexity manageable.

This is especially true in space and aerospace where complex systems need to operate optimally often in harsh and inhospitable environments with a high level of reliability and robustness. In Space and Aerospace Sciences, many applications require the solution of global single and/or multi-objective optimization problems, including mixed variables, multi-modal and non-differentiable quantities.

From global trajectory optimisation to multidisciplinary aircraft and spacecraft design, from planning and scheduling for autonomous vehicles to the synthesis of robust controllers for airplanes or satellites, computational intelligence (CI) techniques have become an important – and in many cases inevitable – tool for tackling these kinds of problems, providing useful and non-intuitive solutions.

UTOPIAE (Uncertainty Treatment and Optimisation in Aerospace Engineering) is a four year Research and Training global network co-ordinated from the Aerospace Centre of Excellence at University of Strathclyde and funded through the European Commission's H2020 programme for four years. Whilst UTOPIAE is working within the domain of Aerospace, these tools and solution can be applied in other contexts.

UTOPIAE is the first training network that addresses the challenge of finding the ideal compromise between enhancing reliability and safety and reducing resource utilisation. UTOPIAE stands upon the shoulders of the existing theoretical and practical developments in the areas of Uncertainty Quantification and Optimisation but progresses beyond the state of the art pushing the boundaries to the edge of what can be computed..

From the control of manufacturing processes to air traffic management, from decision making on multi-phase programmes to space situational awareness, Uncertainty Quantification plays a key role to deliver reliable solutions. At the same time optimised solutions have become a necessity and optimisation is now an essential tool to handle the complexity of our world. Different sectors and communities, deal with uncertainties and optimisation in different forms often equivalent or complementary.

UTOPIAE exploits the intimate relationship between optimisation and UQ to make Optimisation Under Uncertainty (OUU) of complex engineering systems tractable.

2. Optimisation

2.1 Brachistochrone problem definition

Johann Bernoulli, the famous Swiss mathematician mostly known for his studies in fluid dynamics, probability and statistics, posed the problem of the brachistochrone to the readers of Acta Eruditorum in June, 1696

> *I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.*
>
> ***Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.***

Are you ready to be challenged by Bernoulli?

The problem posed by Bernoulli is a classic optimisation problem aimed at computing the path that can get you from point A to point B in the shortest time. Thise kind of problem is solved daily by the GPS in your car. To compute the shortest or the fastest path between two addresses the software is solving a problem of reducing to the minimum the distance in the first case, and reducing to the minimum the time in the second case (this is not always the same thing!!!).

The problem posed by Bernulli is known as Brachistochrone, from the Greek words brakhisto="the shortest" and chronos="time".

History records Newton solving it in the quickest time. Just a few decades earlier Galileo, without the benefit of Calculus, looked at this problem and got the incorrect answer, so don’t feel bad if you struggle too!

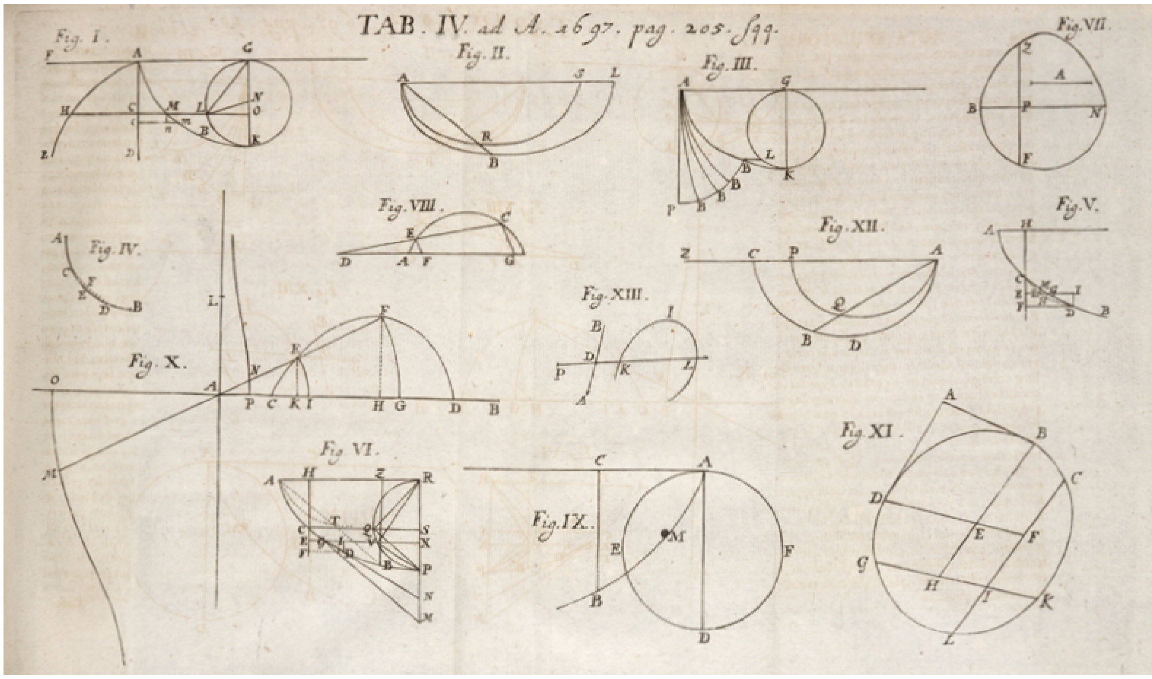


Figure 2.1: Newton attempt to solve the problem

2.2 How to solve the problem mathematically

To calculate the optimal path does not just require vanilla calculus (where you are minimizing a variable in a function), but instead requires minimizing a function that minimizes some other variables. This is known as calculus of variations. The basis of the calculation is about the conservation of energy. The potential energy lost by the falling bead (in the vertical plane), is converted into kinetic energy. If we measure the distance along the arc as s, and an infinitely small piece as ds, then the velocity is

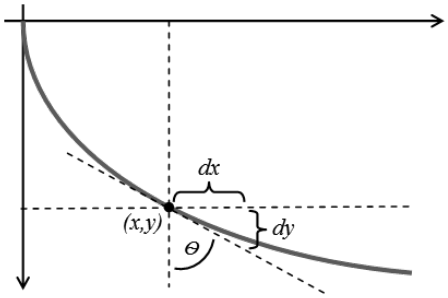


Figure 2.2: Geometrical representation of the problem

$$v = \frac{ds}{dt}$$

and for the conservation of kinetic energy and gravitational potential energy we have that

$$\frac{1}{2}mv^2 = mgy$$

By solving this equation we have that

$$v = \sqrt{2gy}$$

The integral from the time-step when the ball is at the starting location, the origin of the reference system that we call t_A, and the time-step of arrival that we call t_B, is the duration of motion, T computed as

$$T = \int_{t_A}^{t_B} dt$$

By substituting the previous equations we have that

$$T = \int_{t_A}^{t_B} dt = \int_{t_A}^{t_B} \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int_{t_A}^{t_B} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}}$$

where $ds = \sqrt{dx^2 + dy^2}$ is the Pitagora theorem applied to an infinitesimal small step. The path is defined in the graph as $y(x)$, so different paths will have different functions for how the gradient changes. The goal is to find the minimum functional of $y(x)$. Rewriting the last equation as the integral between two points in space x_A, the origin of the system, and x_B the arrival point, we have that

$$T = \frac{1}{\sqrt{2g}} \int_{t_A}^{t_B} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \int_{x_A}^{x_B} \frac{\sqrt{1 + (\frac{dy}{dx})^2}}{\sqrt{x}} dx$$

We know the path is continuous (no gaps, and no instantaneous changes in velocity), and we know there is an acceleration term, so there will be a non-zero second derivative dy^2/dx^2, and we know the two values for the endpoints.

The actual analytic derivation of the function $y(x)$ that minimises T is too advanced to be reported here, so cutting to the chase, here is the result. It's a pair of parametric equations for the x and y coordinates in w.r.t θ, the angle reported in the picture above

$$\begin{cases} x(\theta) = k(\theta - \sin\theta) \\ y(\theta) = k(1 - \cos\theta) \end{cases}$$

Where k is a constant scaled to make sure the curve passes through the end point (x_B, y_B).

The above parametric equations describe a curve called a *cycloid*. A cycloid is usually pictured as the path of a point on a circle that is rolling along the ground like a wheel

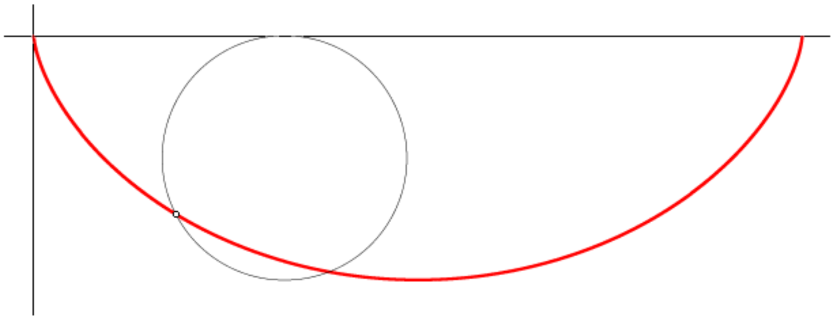


Figure 2.3: Cycloid

2.3 How to solve the problem experimentally

Build a cycloidal track and for comparison purposes also a straight and a highly curved one. To do this, proceed as follows:

- Mark a point on the circumference of a hoop, lid, or other circular object, whose radius you have measured. The height of your brachistocrone ramp is going to be diameter of the circumference you will use.
- Roll it in a vertical plane and trace the locus of the point on a piece of cardboard placed behind the rolling object.
- Transfer the trace to a 2 cm-thick board and cut out very carefully with a jigsaw along the locus of points
- Lay along the profile line a flexible plastic track with a groove, of the same width as the thickness of the board, obtainable from household or electrical supplies stores. An alternative is to use plastic tubes that need to be fixed on the support as in figure 2.4

Your cycloid track is ready.
Building the straight one and the highly curved one are straightforward. 2.4. You can place a target to be hit at the end of the track. The first ball on the track that will hit it is the winner... guess who it is going to be? The cycloid curve of course!

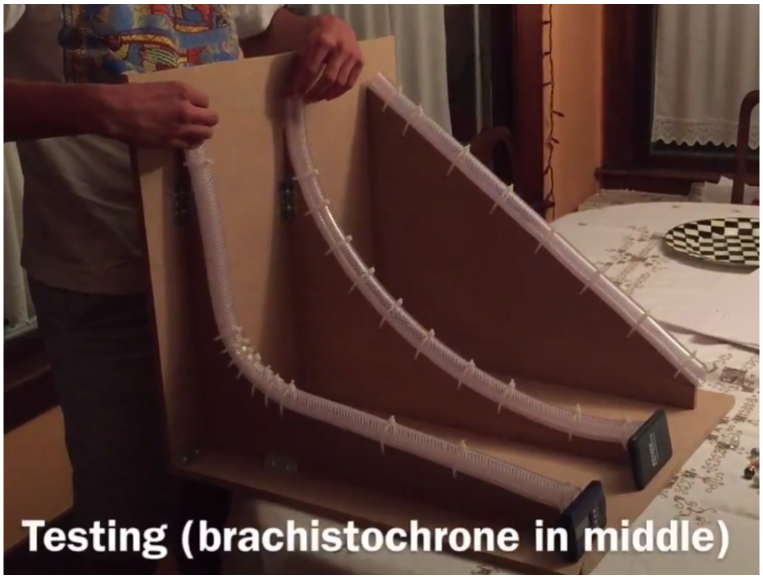


Figure 2.4: Example of racing tracks

Material needed

- 5, 2cm thick board
- 1 circular object (lid, hoop, ...)
- jigsaw
- 3 plastic tracks or 3 plastic tubes
- glue gun and/or electrical clamps
- 3 marble balls and 3 targets

2.4 Examples of Brachistochrone in real life

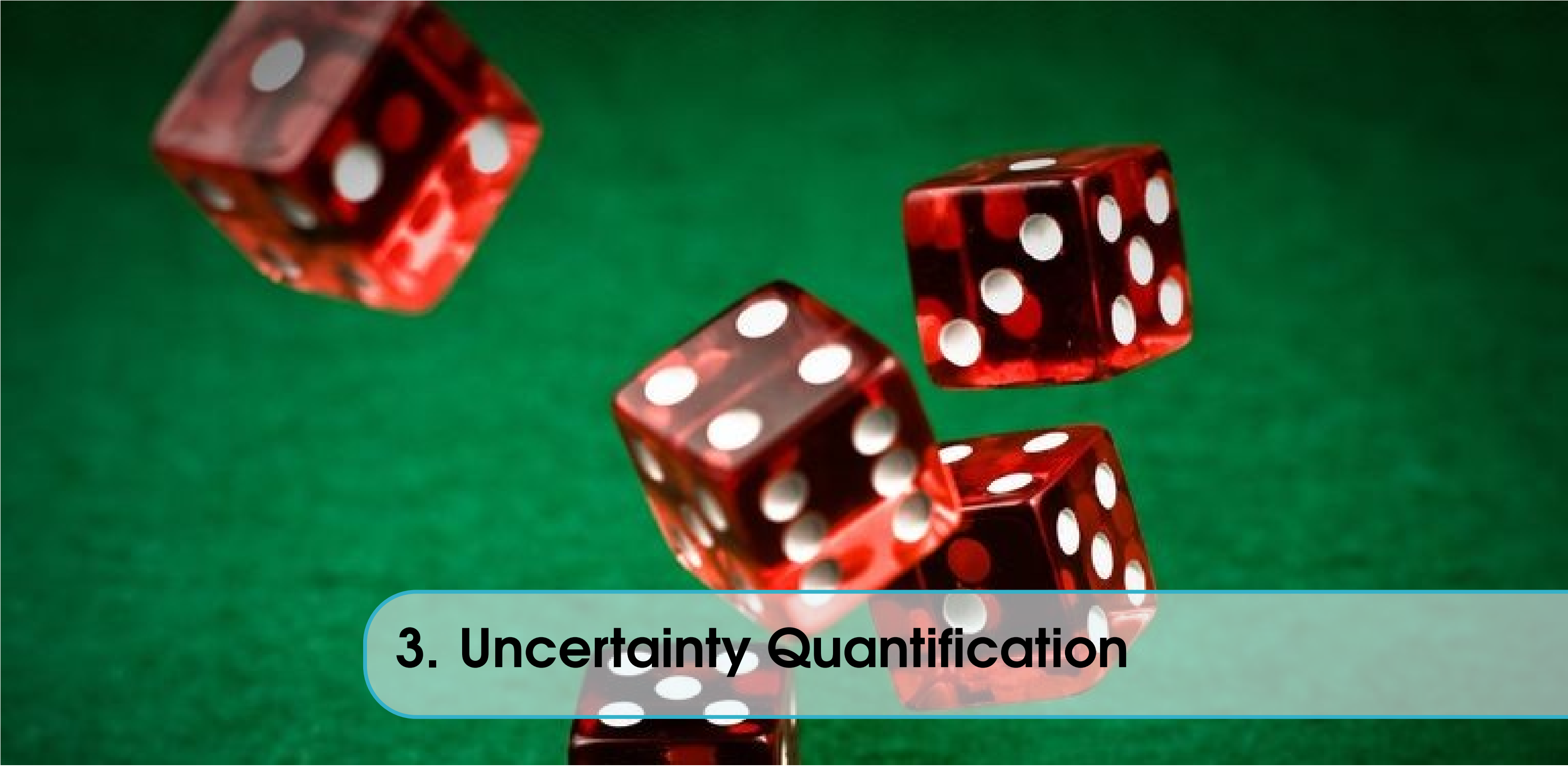
- **Rollercoaster** (Figure 2.5a): Brachistochrone curves are useful for engineers and designers of roller coasters. These people might have a need to accelerate the car to the highest speed possible in the shortest possible vertical drop. As we have just proved, the brachistochrone path is the quickest way to get between two points.
- **Skateboard half pipe** (Figure 2.5b): A mathematically perfect skate park bowl should have sides that are shaped as a tautochrone (a brachistocrone in a uniform gravitational field). If you collide with someone on one of these, you can rest assured that everyone, and their equipment will get to a pile at the bottom of the hill at the same time!
- **Ski jumping** (Figure 2.5c): As I understand it, there is no defined standard for ski-jump profiles (many being built-into the surrounding terrain and using the natural gradient), but if you were going to construct one, you could give it a brachistochronic profile. A bonus of this is that it you could enter the ramp at any point of the curve, and it would take the same amount of time before you flew off the edge. By slowly working your way up the ramp you could gradually increase your exit speed whilst keeping your 'slope time' a constant.
- **Surfing** (Figure 2.5d): many surfing manoeuvres follow the line of the brachistochrone curve whether it is executing a turn down a wave to carve back up and rejoin the peel of a spilling wave or getting up to speed as quickly as possible to ride the barrel of a plunging wave. In fact ,surfing is about having fun and maybe the surfers are simply taking the path which gives them the greatest sense of acceleration

(a) Rollercoaster

(b) Skateboard half pipe

(c) Ski jumping

(d) Surfing



3. Uncertainty Quantification

3.1 Probability and Statistics

Probability is the study of random events. It is used in analyzing games of chance, genetics, weather prediction, and a myriad of other everyday events. **Statistics** is the mathematics we use to collect, organize, and interpret numerical data. It is used to describe and analyze sets of test scores, election results, and shoppers' preferences for particular products. Probability and statistics are closely linked because statistical data are frequently analyzed to see whether conclusions can be drawn legitimately about a particular phenomenon and also to make predictions about future events. For instance, early election results are analyzed to see if they conform to predictions from pre-election polls and also to predict the final outcome of the election.

Understanding probability and statistics is essential in the modern world, where the print and electronic media are full of statistical information and interpretation. The goal of mathematical instruction in this area should be to make students sensible, critical users of probability and statistics, able to apply their processes and principles to real-world problems. Students should not think that those people who did not win the lottery yesterday have a greater chance of winning today! They should not believe an argument merely because various statistics are offered. Rather, they should be able to judge whether the statistics are meaningful and are being used appropriately.

In the area of probability, young children start out simply learning to use probability terms correctly. Words like possibly , probably , and certainly have definite meanings, referring to the increasing likelihood of an event happening, and it takes children some time to begin to use them correctly. Beyond that, though, elementary age children are certainly able to understand the probability of an event . Starting with phrases like once in six tosses , children progress to more sophisticated probability language like chances are one out of six, and finally to standard fractional, decimal, and percent notation for the expression of a probability. To motivate and foster that maturation, students should be regularly engaged in predicting and determining probabilities.

The **theoretical probability** is the probability based on a mathematical analysis of the physical properties and behavior of the objects involved in the event. For instance, when a fair die is rolled each face is equally likely to wind up on top, and so the probability of any particular face showing is one-sixth. **Experimental probabilities** are determined by data gathered through experiments.

The probability of an event is generally computed as

Probability of an event = (number of ways it can happen) / (total number of outcomes)

- The probability of an event can only be between 0 and 1 and can also be written as a percentage.
- The probability of event A is often written as $P(A)$,
- If $P(A) > P(B)$, then event A has a higher chance of occurring than event B.
- If $P(A) = P(B)$ then events A and B are equally likely to occur
- The sum of the probability of all possible events is 1

3.2 Experiment

Divide the class in two groups and provide each group with the set of material outlined below. The question is: if you throw 2 dice together and add the two scores:

1. What is the least possible total score?
2. What is the greatest possible total score?
3. What do you think is the most likely total score?

The common die has 6 faces as in Figure 3.1. Hence the least possible score is obtained by summing the smallest values on the die

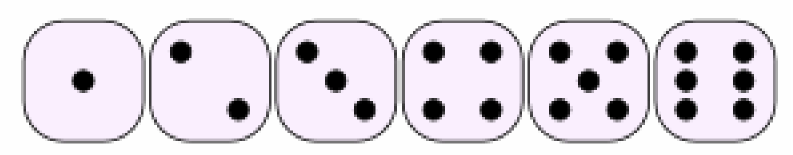


Figure 3.1: Die faces numbered from 1... 6

$$1+1=2$$

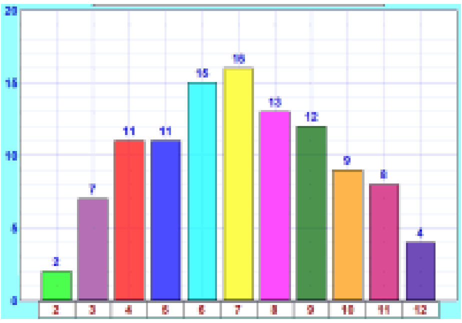
and the greatest possible score is obtained by summing the highest values of the die

$$6+6=12$$

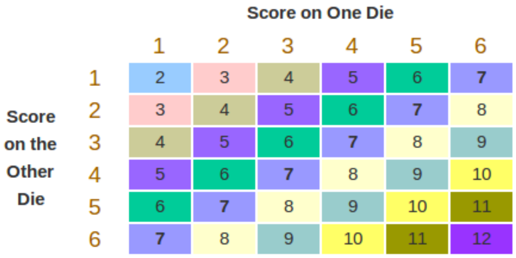
What about the most likely?
To answer this question, each group throws two dice together 108 times, add the scores together each time and record the scores in a table (as in Table 3.1) by using tally marks.
Each then group now draws a bar graph to illustrate their results, something like the one in figure 3.2a. Does the results of the two groups have a similar shape?
So why did you get that shape? The explanation is simple: there is only one way to get a total of 2 (1 + 1), but there are six ways of getting a total of 7 (1 + 6, 2 + 5, 3 + 4, 4 + 3, 5 + 2 and 6 + 1). In Table 3.2b there is the table of all possible outcomes and the total of the sum of the two. Hence there is one way to get 2 and 12, two ways of getting 3 and 11, three ways of getting 4 and 10, ... and so on, till six ways to get a total of 7.
The reason why it has been asked to throw the dice 108 time it is because, given that 36 are the different possible combinations, 108=36x3. Hence theoretically your experimental results should

Added score	Tally marks	Frequency
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
	Total frequency =	108

Table 3.1: Experimental results table



(a) Dice throw results



Score on One Die

Score on the Other Die

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(b) Possible outcomes

look like something as in Figure 3.3 where 2 and 12 happen 3 times, 3 and 11 six times, 4 and 10 nine times, and so on...

How do these theoretical results compare with your experimental results? This graph and your graph should be quite similar, but they are not likely to be exactly the same, as your experiment relied on chance, and the number of times you did it was fairly small. If you did the experiment a very large number of times, you should get results much closer to the theoretical ones. As last exercise you can compute te probability associate to each event:

- The probability of getting 2 is $1/36$
- The probability of getting 3 is $2/36$
- The probability of getting 4 is $3/36$
- ...
-
- The probability of getting 7 is $6/36$
- ...

As final, sum the probability of all events and verify that is equal to 1.

And, by the way, we've now answered the question from near the beginning of the experiment:

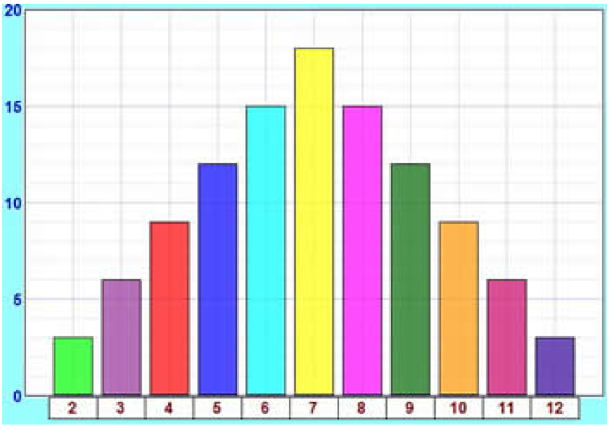


Figure 3.3: Caption

What is the most likely total score?

7 has the highest bar, so 7 is the most likely total score.
Hey, is that why people talk about Lucky 7 ... ?

Material needed

- 2x 2 dice with 6 faces numbered from 1 to 6
- 2x 1 marker
- 2x Pen and paper to record data

3.3 Introduction of Uncertainty

What if the two dice are not standard dice with 6 faces numbered from 1...6? What if the sum is not recorded correctly? These two simple situation will be adding a degree of uncertainty to your experiment.

With the marker the teacher can draw on traditional dice to modify their face number. The experiment above can be repeated with the new set of dice. The teacher will provide the students with the result of the sum of the two dice, the student cannot see the dice themselves. The new results need to be compared with the previous one.

What can you infer from the new results? Are the results you wehere expecting? How can you treat the uncertainty in the problem?